

DOE/ER/61011--T6

IMPROVED MAGNUS' FORM APPROXIMATION OF SATURATION  
VAPOR PRESSURE

by

Oleg A. Alduchov<sup>1</sup> and Robert E. Eskridge

National Climatic Data Center  
National Oceanic and Atmospheric Administration  
Asheville, NC 28801

**DISCLAIMER**

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

MASTER

1 Permanent address: Russian Research Institute of  
Hydrometeorological Information, 6 Korolyov St., Obninsk, Kaluga  
Reg., 249020, Russia

**DISCLAIMER**

**Portions of this document may be illegible  
in electronic image products. Images are  
produced from the best available original  
document.**

## 1. Introduction

Relative humidity is usually measured in aerological observations and dew point depression is usually reported in upper-air reports. These variables must frequently be converted to other moisture variables in meteorological analysis. If relative humidity is converted to vapor pressure, most humidity variables can then be determined. Elliott and Gaffen (1991) reviewed the practices and procedures of the US radiosonde system. In their paper, a comparison of the relative errors was made between the saturation vapor pressure formulations of Tetens (1930), Goff-Gratch (1946), Wexler (1976), and Buck (1981). In this paper, we will expand the analysis of Elliott and Gaffen by deriving several new saturation vapor pressure formulas, and reviewing the various errors in these formulations. We will show that two of the new formulations of vapor pressure over water and ice are superior to existing formulas.

Upper air temperature data are found to vary from about +50 C to -80 C. This large variation requires a saturation vapor pressure equation to be accurate over a large temperature range. While the errors introduced by the use of relatively inaccurate conversion equations are smaller than the errors due to the instruments, dewpoint coding errors, and dewpoint conversion algorithms (Elliott and Gaffen, 1993); they introduce additional systematic errors in humidity data.

The most precise formulation of vapor pressure over a plane surface of water was given by Wexler (1976). The relative errors of Tetens' (1930) formula and one due to Buck (1981) (Buck's equation is recommended in the Federal Meteorological Handbook No. 3, 1991) are shown in Table I. The relative errors in this table are the predicted value minus the Wexler value divided by the Wexler value. Table I shows that in a standard atmosphere at low temperatures the relative errors are large and, more importantly, systematic. The magnitude and distribution of the relative errors are different for different approximations.

Humidity data that exists in publications or databases have frequently been converted using formulations of saturation vapor pressure based on various fits to the data (such as Wexler (1976), Goff and Gratch (1946), and Sonntag (1990)). In order to use older data together with new data, it is desirable to develop new approximations which are simple to use and minimize the difference between these methods. The purpose of this research is to find Magnus' form approximations of saturation vapor pressure which are close to the three prominent formulations of saturation vapor pressure, and accurate.

## 2. Definitions

Pure water vapor in a state of equilibrium with a plane surface of pure water or ice at the same pressure and temperature is said to be saturated. The saturation vapor pressure of pure water vapor is a function of temperature alone (Clausius-

**Table I.** Relative errors(%) of Buck's (BU81) and Tetens' (TE30) approximations.

P hPa	t °C	BU81	TE30
1000	14.3	.468	.394
900	8.6	.380	.328
850	5.5	.291	.262
700	-4.5	.208	.318
500	-21.2	.373	.988
400	-31.7	.847	2.007
300	-43.9	2.044	4.130

Clausius equation). The saturation vapor pressure of pure water vapor is denoted by  $E_w$  or  $E_i$  for saturation with respect to water or ice, respectively.

When moist air exists in a state of equilibrium with a surface of pure water or ice, it is said to be saturated. The saturation vapor pressure of moist air is denoted by  $E_{w_s}$  and  $E_{i_s}$  for saturation with respect to water and ice, respectively.

A correction factor,  $F$ , is defined which accounts for the departure of the mixture of air and water vapor from the ideal gas laws, and it is given by

$$F_w = \frac{E_{wa}}{E_w}, \quad F_i = \frac{E_{ia}}{E_i}. \quad (1)$$

In this paper, capital E's will refer to basic formulations of vapor pressure (Wexler, Goff and Gratch, and Sonntag), and small e's will refer to approximations to the basic vapor pressure formulations (e.g. Buck). Capital T is temperature in degrees Kelvin, small t is temperature in degrees Celsius, and pressure (p, e, and E) is in units of hPa (mbs).

Calculations of humidity in meteorology are based on using the saturation vapor pressure over a surface of water or ice, and this calculation depends only on temperature. Since the undercooling of water below its melting point is common in clouds, the WMO (1966) recommends that vapor pressure over a plane of water be used for calculation of humidity variables for negative temperatures.

### 3. Basic formulations of vapor pressure

The first accurate formula for vapor pressure was developed by Goff and Gratch (1945) (we denote this paper by GG45):

$$\begin{aligned} \ln E_w = & 7.90298(1-T_s/T) + 5.02808 \ln (T_s/T) \\ & + 1.3816 \times 10^{-7} (1-10^{11.344(1-T/T_s)}) \\ & - 8.1328 \times 10^{-3} (1-10^{-3.49149(1-T_s/T)}) \\ & + \ln E_{w0}, \end{aligned} \quad (2)$$

where  $E_w$  - saturation vapor pressure in hPa with respect to a plane surface of pure water;

$T$  - temperature in K;

$T_s$  - temperature at the steam point (=373.16 K);

$E_{w0}$  - saturation vapor pressure in hPa at the steam point temperature (=1013.246 hPa).

This formulation was soon modified by Goff and Gratch (1946) (we denote this paper by GG46):

$$\begin{aligned} \ln E_w = & 10.79574(1-T_1/T) - 5.02800 \ln (T/T_1) \\ & + 1.50475 \times 10^{-4} (1-10^{8.2969(1-T/T_1)}) \\ & - 0.42873 \times 10^{-3} (1-10^{4.76955(1-T_1/T)}) \\ & + 0.78614, \end{aligned} \quad (3)$$

where  $T_1$  is the temperature at the triple point of water (273.16 K).

This second formulation was significantly different from (2), the difference being more than 5%. Equation (3) was recommended by the WMO in 1966 for use in meteorological calculations and is still in use in many countries. The Smithsonian meteorological tables (List, 1949) are based on this equation.

Goff (1965) revised the formulation, denoted as GG65, based on a new analysis:

$$\begin{aligned} \ln E_w = & 10.79586(1-T_1/T) - 5.02808 \ln (T/T_1) \\ & + 1.50474 \times 10^{-4} (1-10^{8.29692(1-T/T_1)}) \\ & - 0.42873 \times 10^{-3} (1-10^{4.76955(1-T_1/T)}) \\ & - 2.2195983 + \lg E_{w1}, \end{aligned} \quad (4)$$

where  $E_{w1}$  is the saturation vapor pressure in hPa at the steam point temperature (1013.246 hPa).

But this approximation, (4), is not significantly different from equation (3) developed in 1946. The largest relative

difference between the GG65 and the GG46 formulations is 0.008 percent at -40 C between -40 C and +50 C. The GG65 and GG46 equations are essentially identical.

In 1976, Wexler published a new vapor pressure formulation which was based on new measurements (Stimson, 1969; Guildner et al., 1976) and a new value of the gas constant (Cohen and Taylor, 1973). We denote this formulation as WE76:

$$E_w = 0.01 \exp(-2.9912729 \times 10^3 T^{-2} - 6.0170128 \times 10^3 T^{-1} + 1.887643845 \times 10^1 - 2.8354721 \times 10^{-2} T + 1.7838301 \times 10^{-5} T^2 - 8.4150417 \times 10^{-10} T^3 + 4.4412543 \times 10^{-13} T^4 + 2.858487 \ln T). \quad (5)$$

The values of saturation vapor pressure from Wexler's formulation, (5), are markedly different from Goff and Gratch's (1946) equations at negative temperatures. The largest relative difference between these formulations is 0.700 percent for temperatures between -40 C and 0 C (at -40 C) and 0.045 percent for temperatures between 0 and +50 C (at 0 C).

Sonntag and his colleague Heinze have conducted several important investigations of the relationship between vapor pressure and temperature. In 1982, Sonntag and Heinze published equations for saturation vapor pressure based on the "International Temperature Scale 1968" (S082):

$$E_w = \exp(-6094.4642 T_{68}^{-1} + 16.519825 - 2.7245552 \times 10^{-2} T_{68} + 1.6853396 \times 10^{-5} T_{68}^2 + 2.433502 \ln T_{68}), \quad (6)$$

where  $T_{68}$  is the temperature in K in Temperature Scale 1968. In 1990, Sonntag published a formulation, denoted as S090, of vapor pressure based on the new "International Practical Temperature Scale 1990":

$$E_w = \exp(-6096.9385 T_{90}^{-1} + 16.635794 - 2.711193 \times 10^{-2} T_{90} + 1.673952 \times 10^{-5} T_{90}^2 + 2.433502 \ln T_{90}), \quad (7)$$

where  $T_{90}$  is the temperature in K in Temperature Scale 1990.

Saturation vapor pressures calculated from (6) and (7) are not significantly different from Wexler's formulation, equation (5). The largest relative difference between Wexler's formulation and Sonntag's 1982 formulation is 0.055 percent at 50 C, and the largest difference between the 1990 formulation and Wexler's is 0.120 percent at 50 C.

Figure 1 shows the relative differences between the S082,

S090, GG45, and the WE76 formulations with respect to the GG46 formulation.

As we have pointed out, there are three basic formulations for calculating saturation vapor pressure. First, the most widely used and approved by the WMO for use in meteorological practice is the GG46 formulation; second; the more recent, and (we believe) more accurate WE76 formulation which is also widely used; and finally, the S090 formulation which differs only slightly from the WE76 formulation.

An ideal saturation vapor pressure approximation will minimize the errors between it and the three basic formulations (GG46, WE76, and S090). Such an approximation will be highly accurate, and it will minimize the disruption due to the introduction of a new equation.

#### 4. Approximations of vapor pressure over water

Vapor pressure formulations WE76, GG46, and S090 are very complex and inconvenient. This complexity has led to various simplified and less accurate approximations of vapor pressure. Good reviews of these approximations can be founded in Sargent (1980), Abbott and Tabony (1985), Alduchov (1988), and Elliott and Gaffen (1991). Some of the most widely applicable approximations are shown in Table II.

The most convenient form, equation (8), is due to Magnus (1844)

$$E_w = c e^{at/(b+t)}, \quad (8)$$

or

$$E_w = c 10^{a_1 t/(b+t)}, \quad (9)$$

where  $t$  is the temperature in C and

$$a = a_1 \ln 10. \quad (10)$$

The main reason to derive approximations to the basic formulations is simplicity; it is easier to convert between temperature and the saturation vapor pressure with minimal error. The requirements of simplicity and minimal error will eliminate many of the approximations. It is difficult to suggest a more convenient form for the saturation vapor pressure than the

Table II. Saturation vapor pressure approximations  
above a plane of water

Author	Name	Formula
<i>Magnus-formulas:</i>		
Tetens(1930)	TE30	$e(t) = 6.11 \times 10^{7.5} t / (237.3 + t)$
Matveev(1967)	MA67	$e(t) = 6.1078 \times 10^{7.63} t / (241.9 + t)$
Buck-1(1981)	BU81	$e(t) = 6.1121 \times e^{17.502 t / (240.97 + t)}$
Abbott-Tabony(1985)	AT85	$e(t) = 6.1070 \times e^{17.38 t / (239.0 + t)}$
Alduchov(1988)	AL88	$e(t) = 6.1070 \times 10^{7.665 t / (243.33 + t)}$
Sonntag(1990)	SA90	$e(t) = 6.112 \times e^{17.62 t / (243.12)}$
<i>Non Magnus-formulas:</i>		
Tabata(1973)	TA73	$e(T) = 10^{8.42926609 - 1927.17843 / T - 71208.271 / T^2}$
Buck-2(1981)	BU-2	$e(t) = 6.1121 \times e^{t(18.729 - t / 227.3) / (t + 257.87)}$
Hooper(1986)	HO86	$e(t) = (1.3521 + 1.6369 \times 10^{-2} t + 3.1794 \times 10^{-5} t^2 - 1.4892 \times 10^{-7} t^3)^6$
<i>Approximation developed this research:</i>		
	AEKD	$e(t) = 6.1102 \times e^{17.621 t / (242.97 + t)}$
	AEGD	$e(t) = 6.1105 \times e^{17.546 t / (241.81 + t)}$
	AEWD	$e(t) = 6.1128 \times e^{17.610 t / (242.89 + t)}$
	AESD	$e(t) = 6.1152 \times e^{17.616 t / (242.91 + t)}$
	AEKR	$e(t) = 6.1094 \times e^{17.625 t / (243.04 + t)}$
	AEGR	$e(t) = 6.1072 \times e^{17.578 t / (242.25 + t)}$
	AEWR	$e(t) = 6.1085 \times e^{17.654 t / (243.49 + t)}$
	AESR	$e(t) = 6.1107 \times e^{17.660 t / (243.51 + t)}$

Magnus' form.

The motivation for the appearance of new approximations for saturation vapor pressure approximation is insufficient accuracy when the Magnus' formula is used with published coefficients (a, b, c). However, as shown below, it is possible to develop highly accurate vapor pressure formulations using the Magnus' form. We shall find coefficients a, b and c which will accurately approximate all three basic formulations of the saturation vapor pressure.

To determine the accuracy of a vapor pressure approximation,  $e(t)$ , we need a criterion of accuracy. We shall assume that the observation error of the thermistor is zero, and the temperature error is due only to roundoff. It is standard meteorological practice to report temperature to a tenth of a degree. Therefore, the roundoff error in temperature will be less than or equal to 0.05 degree. The error in the vapor pressure due to this temperature error in any basic formulation, E, is

$$dE(t) = \text{MAX}_{-0.05 \leq \tau \leq 0.05} |E(t) - E(t+\tau)| \quad (11)$$

We will say that a particular approximation,  $e(t)$ , is



accurate, if the difference between this approximation and a corresponding basic formulation is not more than the error in vapor pressure due to the temperature roundoff error. Hence, the measure of accuracy for approximation  $e(t)$  is

$$d = \underset{-40 \leq t \leq 50}{\text{MAX}} \left| \frac{e(t) - E(t)}{dE(t)} \right|. \quad (12)$$

The approximation  $e(t)$  is sufficiently accurate if  $d \leq 1$ . For example,  $d \leq 1$ , means that the relative error with respect to the GG46 formulation is less than or equal to 0.52 % at -40 C and 0.25 % at 50 C. This definition of accuracy, (12), leads to almost identical limits of relative error for the WE76 formulation.

We determined the accuracy of each approximation,  $e$ , with respect to the three basic formulations of vapor pressure (GG46, WE76, and S090). We denote the measures of accuracy from equation (12) as  $d_g$ ,  $d_w$ , and  $d_s$ , respectively. The measure of accuracy with respect to all three formulations,  $d_k$ , is defined as

$$d_k = \underset{a, b, c}{\text{MAX}} (d_g, d_w, d_s). \quad (13)$$

We will denote the maximum relative errors for these approximations by  $r_g$ ,  $r_w$ ,  $r_s$ , respectively. We define  $r_k$  as

$$r_k = \text{MAX} (r_g, r_w, r_s). \quad (14)$$

Using these definitions of error, we can find coefficients  $a$ ,  $b$ , and  $c$  for the Magnus' formula (8) which minimize  $r_g$  for the approximations AEGR, minimize  $r_w$  for AEWR, minimize  $r_s$  for AESR, and minimize  $r_k$  for AEKR (see Table II).

The techniques used to develop the Magnus' form equations are given in the appendix. We have chosen to use the minimum of the maximum deviation rather than, say, minimizing the total deviation over the range or a least squares fit, because it will not permit the equation to have any large magnitude deviations from the basic formulation. A least squares fit or a minimum of the maximum can allow large deviations from the basic formulation.

Table III. Maxima of relative errors ( $r\%$ ) and measures of accuracy ( $d$ ) for various basic formulations and approximations of saturation vapor pressure over water. Temperatures is  $-40$  to  $50\text{C}$ . Temperature in C at the maximum is given in brackets.

	$r_g(t)$	$d_g(t)$	$r_w(t)$	$d_w(t)$	$r_s(t)$	$d_s(t)$	$r_k(t)$	$d_k(t)$
GG46	.000(-)	.000(-)	.695(-40)	1.343(-40)	.648(-40)	1.252(-40)	.695(-40)	1.343(-40)
WE76	.700(-40)	1.346(-40)	.000(-)	.000(-)	.120(50)	.482(50)	.700(-40)	1.346(-40)
SO90	.653(-40)	1.255(-40)	.120(50)	.482(50)	.000(-)	.000(-)	.653(-40)	1.255(-40)
GG45	5.109(-40)	9.825(-40)	5.768(-40)	11.145(-40)	5.724(-40)	11.057(-40)	5.768(-40)	11.145(-40)
GG65	.008(-40)	.016(-40)	.703(-40)	1.359(-40)	.657(-40)	1.269(-40)	.703(-40)	1.358(-40)
SO68	.732(-40)	1.407(-40)	.055(50)	.223(50)	.078(-40)	.260(50)	.732(50)	1.407(-40)
TE30	2.537(-40)	4.879(-40)	3.214(-40)	6.210(-40)	3.169(-40)	6.121(-40)	3.214(-40)	6.210(-40)
MA67	.558(-40)	1.448(50)	1.249(-40)	2.413(-40)	1.202(-40)	2.323(-40)	1.249(-40)	2.413(-40)
BU81	.769(-40)	1.478(-40)	1.458(-40)	2.817(-40)	1.412(-40)	2.727(-40)	1.458(-40)	2.817(-40)
AT85	1.834(-40)	3.527(-40)	2.516(-40)	4.862(-40)	2.471(-40)	4.772(-40)	2.516(-40)	4.862(-40)
AL88	.341(-31)	1.018(50)	.406(-40)	1.053(50)	.360(-40)	.832(24)	.406(-40)	1.053(50)
SA90	.597(-35)	1.225(-31)	.292(-23)	.822(28)	.328(28)	1.135(30)	.597(-35)	1.225(-31)
BU-2	.493(-40)	.948(-40)	.205(-40)	.398(50)	.159(-40)	.306(-40)	.493(-40)	.948(-40)
TA73	1.322(-40)	2.542(-40)	.618(-40)	1.716(50)	.665(-40)	1.292(-38)	1.322(-40)	2.542(-40)
HO86	.140(-28)	.306(-25)	.592(-40)	1.143(-40)	.545(-40)	1.053(-40)	.545(-40)	1.143(-40)
Newly developed approximations:								
AEKD	.373(-29)	.804(-26)	.412(-40)	.806(50)	.365(-40)	.804(27)	.412(-40)	.806(50)
AEGD	.220(-40)	.430(50)	.913(-40)	1.764(-40)	.866(-40)	1.674(-40)	.913(-40)	1.764(-40)
AESD	.477(-31)	1.018(-27)	.290(-40)	.561(27)	.252(27)	.862(29)	.477(-31)	1.018(-27)
AEWD	.456(-29)	1.006(50)	.334(-40)	1.040(50)	.288(-40)	.563(27)	.456(-29)	1.040(50)
AEKR	.384(-30)	.822(-27)	.384(-40)	.742(-40)	.337(-40)	.868(27)	.384(-40)	.868(27)
AAGR	.151(24)	.594(50)	.844(-40)	1.630(-40)	.797(-40)	1.540(-40)	.844(-40)	1.630(-40)
AEWR	.508(-36)	1.033(-32)	.203(-22)	.812(50)	.278(24)	.932(26)	.508(-36)	1.033(-32)
AESR	.471(-34)	1.253(50)	.320(50)	1.288(50)	.202(-40)	.804(50)	.471(-34)	1.288(50)

We have found several approximations which do not minimize the relative errors, but maximize accuracy. These approximations are: AEGD (min  $d_g$ ), AEWD (min  $d_w$ ), AESD (min  $d_s$ ), and AEKD (min  $d_k$ ). See Table II for the equations.

Table III shows the relative errors and measure of accuracy for the new approximations, and commonly used approximations. Note that the maximum relative errors usually occur at the end points of the temperature range ( $-40\text{C}$ ,  $50\text{C}$ ). Table III shows that only three of the approximations satisfy the criterion for accuracy,  $d_k \leq 1$ . They are BU-2, an approximation of Buck (1981), and two approximations we have found (AEKR and AEKD). The AEKR and AEKD approximations have the Magnus' form. These two approximations maximize accuracy and minimize errors, in that  $d_k$  and  $r_k$  are smallest for these approximations.

Fig. 2 shows part of the surface  $d_k(a,b,c)$  in the neighborhood of the minimum for approximation AEKD ( $c=6.1102$ ). The coefficients  $a$  and  $b$  for several Magnus' form approximations are shown. Clearly, this surface is complex, and it has several local minimums. Fig. 2b is an enlargement of the rectangular area of Fig 2a. The hashed area contains the region in which coefficients  $a$  and  $b$  satisfy the requirement that  $d_k \leq 1$ , and  $c = 6.1102$ .

Fig.3 shows the behavior of several vapor pressure approximations with respect to the Wexler formulation for temperatures between -40 C and 50 C. Fig. 3 shows that several of the approximations are excellent over the temperature range of 0 C to 40 C, and that the AEKR Magnus' formulation is the most accurate over the entire -40 C to 50 C temperature range.

When we extend the lower limit of the temperature range from -40 to -80 C, we will find that differences between the basic formulations increase. For example, the relative difference between the GG46 and WE76 formulations is more than 10% at -80 C with respect to WE76. Obviously it will be impossible to approximate all three formulations with a relative error of less than 5%. The best way to find an accurate approximation is to choose the basic formulation to use and develop an approximation for this formulation. We have found the three best, minimum relative errors, approximations for the temperature range from +50 to -80 C:

$$\begin{aligned}
 AEGR_{50,-80}(t) &= 6.1037 e^{17.641t/(243.27+t)}, & (0.337/0.823) \\
 AEW_{50,-80}(t) &= 6.0612 e^{18.102t/(249.52+t)}, & (0.852/3.429) \\
 AESR_{50,-80}(t) &= 6.0620 e^{18.112t/(249.59+t)}, & (0.845/3.386)
 \end{aligned}$$

In brackets are given the maximum relative errors and measure of accuracy for the corresponding approximation. For comparison we show in Table IV the relative errors and measure of accuracy for three approximations with respect to the three basic formulations for temperature range from 50 to -80 C.

Table IV. Maximum relative errors and measure of accuracy (below) for the BU81, SA90 and AEKR approximations of saturation vapor pressure with respect to the basic formulations for temperatures between -80 and 50 C.

	GG46	WE76	SO90
BU81	4.850/ 6.005	14.423/18.499	14.312/18.355
SA90	0.707/ 1.252	9.425/10.089	9.308/11.937
AEKR	0.539/ 0.822	10.067/12.067	9.951/12.767

##### 5. Approximations of vapor pressure over ice.

There are also three main models for calculating saturation vapor pressure over a plane surface of ice. The model equations are valid for the temperature range of -80 C to 0 C. The first accurate model (GG46i) was developed by Goff and Gratch (1946),

and it is given by

$$\ln E_i = 9.096936(1-T_1/T) - 3.56654 \ln(T_1/T) + 0.876817(1-T/T_0) - 2.2195983 + \ln E_{w_0} \quad (15)$$

where:  $E_i$  is the saturation vapor pressure in hPa with respect to a plane surface of ice;  
 $T$  is the temperature;  
 $T_1$  is the temperature at the triple-point of water;  
 $E_{w_0}$  is the saturation vapor pressure at the steam point temperature.

Wexler's (1976) formulation (WE76i) is given by

$$E_i = 0.01 \exp(-5.8653696 \times 10^3 T^{-1} + 2.2241033 \times 10 + 1.3749042 \times 10^{-2} T - 3.4031775 \times 10^{-5} T^2 + 2.6967687 \times 10^{-7} T^3 + 0.6918651 \ln T), \quad (16)$$

and Sonntag's formulation (SO90i) is given by

$$E_i = \exp(-6024.5282^3 T_{90}^{-1} + 24.72219 - 1.0613868 \times 10^{-2} T_{90} + 1.3198825 \times 10^{-5} T_{90}^2 + 0.49382577 \ln T_{90}). \quad (17)$$

We have derived coefficients  $a$ ,  $b$  and  $c$  for the general Magnus' formula (8), which minimize  $d_{gi}$ ,  $d_{wi}$ ,  $d_{si}$  and  $d_{ki}$ , over the temperature range of -80 C to 0 C with respect to the basic formulations; GG46i, WE76i, and SO90i. Applying equations (12) and (13) to the basic formulations, we define measures of accuracy:  $\bar{d}_{gi}$ ,  $\bar{d}_{wi}$ ,  $\bar{d}_{si}$ , and  $\bar{d}_{ki}$ . The equations for these new approximations are given in Table V.

We have found new approximations of the basic formulations of saturation vapor pressure over ice which maximize accuracy. These new approximations are AEGDi, AEWDi, AESDi, and AEKRi. Table IV gives the new approximations and Table V shows their relative error and measure of accuracy. Several approximations, which are commonly used to calculate saturation vapor pressure over ice, are also given in Table V.

Magnus' form equations for calculating saturation vapor pressure over ice were also found, which minimized the relative error. These new equations, which are given in Table V, are denoted by AEKRi, AEGri, AEWRi, and AESRi.

Table VI shows the relative errors,  $r$ , and measures of accuracy,  $d$ , for several approximations of saturation vapor pressure over ice found in the literature. The relative errors and measures of accuracy are given for the new relationships we

Table V. Saturation vapor pressure approximations above a plane of ice

Author	Name	Formula
<i>Magnus-formulas:</i>		
Matveev(1967)	MA67i	$e(t) = 6.1078 \times 10^{9.5} t / (265.5 + t)$
Buck-1(1981)	BU81i	$e(t) = 6.1115 \times e^{22.452 t / (272.55 + t)}$
Abbott-Tabony(1985)	AT85i	$e(t) = 6.1070 \times e^{22.44 t / (272.4 + t)}$
Alduchov(1988)	AL88i	$e(t) = 6.1064 \times 10^{9.800 t / (273.66 + t)}$
Sonntag(1990)	SA90i	$e(t) = 6.112 \times e^{22.46 t / (272.62)}$
<i>Non Magnus-formulas:</i>		
Buck-2(1981)	BU-2i	$e(t) = 6.1115 \times e^{t(23.036 - t/333.7) / (t+279.82)}$
Hooper(1986)	HO86i	$e(t) = (1.2539 + 1.2924 \times 10^{-2} t + 1.9187 \times 10^{-5} t^2 - 9.41 \times 10^{-8} t^3)^8$
<i>Newly developed approximations:</i>		
	AEKDi	$e(t) = 6.1128 \times e^{22.571 t / (273.71 + t)}$
	AEGDi	$e(t) = 6.1093 \times e^{22.570 t / (273.69 + t)}$
	AEWDi	$e(t) = 6.1150 \times e^{22.561 t / (273.64 + t)}$
	AESDi	$e(t) = 6.1148 \times e^{22.574 t / (273.72 + t)}$
	AEKRi	$e(t) = 6.1121 \times e^{22.587 t / (273.86 + t)}$
	AEGRi	$e(t) = 6.1084 \times e^{22.575 t / (273.74 + t)}$
	AEWRi	$e(t) = 6.1162 \times e^{22.577 t / (273.78 + t)}$
	AESRi	$e(t) = 6.1140 \times e^{22.579 t / (273.81 + t)}$

Table VI. Maximums of relative errors (r %) and measures of accuracy (d) for some formulations and approximations of vapor pressure over ice for temperatures between -80 and 0 degree C. (Temperature in C for these maximums is given in brackets).

	$r_g(t)$	$d_g(t)$	$r_w(t)$	$d_w(t)$	$r_s(t)$	$d_s(t)$	$r_k(t)$	$d_k(t)$
GG46i	.000(-)	.000(-)	.230(-80)	.294(-48)	.089(-8)	.207(-3)	.230(-80)	.294(-48)
WE76i	.231(-80)	.294(-48)	.000(-)	.000(-)	.156(-80)	.196(-56)	.231(-80)	.294(-48)
SO90i	.089(-8)	.208(-3)	.156(-80)	.196(-56)	.000(-)	.000(-)	.156(-80)	.208(-3)
MA67i	10.672(-80)	12.930(-80)	10.878(-80)	13.180(-80)	10.738(-80)	13.010(-80)	10.878(-80)	13.180(-80)
BU81i	.673(-80)	.816(-80)	.902(-80)	1.093(-80)	.747(-80)	.905(-80)	.902(-80)	1.013(-80)
AT85i	.973(-80)	1.179(-80)	1.201(-80)	1.455(-80)	1.047(-80)	1.268(-80)	1.201(-80)	1.455(-80)
AL88i	.094(-80)	.189(-17)	.324(-80)	.456(-19)	.174(-17)	.375(-14)	.324(-80)	.456(-19)
SA90i	.672(-80)	.814(-80)	.900(-80)	1.091(-80)	.745(-80)	.903(-80)	.900(-80)	.903(-80)
B281i	.284(-80)	.345(-80)	.054(-80)	.065(-80)	.210(-80)	.254(-80)	.284(-80)	.345(-80)
HO86i	.193(-80)	.234(-80)	.423(-80)	.512(-80)	.267(-80)	.324(-80)	.423(-80)	.512(-80)
<i>Newly developed approximations:</i>								
AEKDi	.171(-62)	.256(0)	.213(-80)	.258(-21)	.106(-61)	.166(-16)	.213(-80)	.258(-21)
AEGDi	.095(-80)	.120(-59)	.325(-80)	.394(-80)	.169(-80)	.299(-16)	.325(-80)	.394(-80)
AEWDi	.282(-63)	.412(-59)	.101(-80)	.122(-80)	.216(-62)	.317(-59)	.282(-63)	.412(-59)
AESDi	.148(-60)	.335(0)	.256(-80)	.310(-80)	.100(-80)	.129(0)	.256(-80)	.335(0)
AEKRi	.163(-65)	.231(-63)	.164(-80)	.327(-22)	.109(-19)	.230(-17)	.164(-80)	.327(-22)
AEGRi	.076(-20)	.159(-19)	.306(-80)	.429(-21)	.160(-19)	.339(-16)	.306(-80)	.429(-16)
AEWRi	.272(-65)	.391(0)	.069(-63)	.153(0)	.205(-64)	.295(-61)	.272(-65)	.391(0)
AESRi	.276(-67)	.389(-63)	.098(-22)	.202(-20)	.207(-66)	.294(-63)	.276(-67)	.389(-63)



have found. The two most accurate approximations are AEKDi and AEKRi. Table VI shows that the relative error for AEKDi and AEKRi are 0.213 and 0.164%, respectively. Also the accuracy,  $d$ , of AEKDi and AEKRi are 0.258 and 0.327%, respectively. Both of these formulations are more accurate than the BU81i formulation (Buck, 1981).

## 6. Enhancement factor

The departure of the ideal gas law for a mixture of air and water vapor leads to errors, which can be eliminated by an enhancement or correction factor. The enhancement or correction factor is defined as the ratio of the saturation vapor pressure for moist air to that of pure water vapor over a plane of water

$$F_w(t, p) = E_{wa}(t, p) / E_w(t), \quad (18)$$

and over a plane surface of ice

$$F_i(t, p) = E_{ia}(t, p) / E_i(t). \quad (19)$$

Neglecting the correction factor results in relative errors (calculated-observed/observed) in the saturation vapor pressure of moist air which can reach 0.596 % with respect to water (at -40 C, 1000 hPa) and 0.882 % with respect to ice (at -80 C, 1000 hPa).

We have developed the following approximations to the data in Table 4.10 WMO (1966), which describes the behavior of the enhancement factor over water and ice:

$$f_w(p) = 1.00071 \times e^{0.0000045p}, \quad (20)$$

and

$$f_i(p) = 0.99882 \times e^{0.000008p}, \quad (21)$$

where  $p$  is the pressure in hPa. These two approximations have maximum relative errors of 0.0773 % (at 0 C, 900 hPa) over a plane surface of water, and 0.209% relative error (at -80 C, 1000 hPa) over a plane surface of ice, respectively.

Buck (1981) developed equations for the enhancement factor for water and ice. Buck' equations are:

$$f_w = 1.0007 + 3.46 \times 10^{-6} p, \quad (22)$$

and

$$f_i = 1.0003 + 4.18 \times 10^{-6} p. \quad (23)$$

Equations (22) and (23) have maximum relative errors of 0.183 % and 0.438 %, respectively. The new approximations (21) and (22) of the enhancement coefficient are more accurate than Bucks' approximations.

Bögel (1977) developed the most accurate fit to experimental data for the enhancement factor. The enhancement factor (22) for water gives us a maximum relative error 0.088 % with respect to the enhancement factor for water of Bögel, which is given below:

$$f_w(p, t) = 1 + \frac{10^{-4} e_w(t)}{(273+t)} [(38+173e^{-t/43})(1-e_w(t)/p) - (6.339+4.28e^{-t/107})(1-p/e_w(t))]. \quad (24)$$

## 7. Conclusions

To calculate the saturation vapor pressure of pure water over a plane surface of water for the temperature range of -40 C to 50 C, we recommend the simple and accurate approximation, AEKR:

$$e_w(t) = 6.1094 \times e^{17.625t/(243.04+t)}. \quad (25)$$

This equation gives a maximum relative error of less than 0.384%, and a maximum measure of accuracy of less than 0.868 with respect to any of the three basic formulations GG46, WE76, and SO90.

For a mixture of pure water and moist air, we recommend

$$e_{wa}(t) = 1.00071 \times e^{0.0000045p} \times e_w(t). \quad (26)$$

Equation (26) gives a maximum relative error of less than 0.414%, and a maximum measure of accuracy less than 0.932 with respect to the three basic formulations and the WMO's form (1966) or Bögel's (1977) enhancement factor.

For saturation vapor pressure over a plane surface of ice (-80 C to 0.0 C), we recommend following two approximations. To calculate the saturation vapor pressure of pure water vapor over ice use AEKRi

which has a maximum relative error less than 0.213%, and a

$$e_i(t) = 6.1121 \times e^{22.587 t / (273.86 + t)}, \quad (27)$$

maximum measure of accuracy of less than 0.258 with respect to any from three basic formulations GG46, WE76 or SO90. For moist air use the following equation

$$e_{wi}(t) = 0.99882 \times e^{0.000008 p} \times e_i(t). \quad (28)$$

This equation (28) has a maximum relative error less than 0.397%, and a maximum measure of accuracy less than 0.721 with respect to any from three basic formulations and the WMO (1966) enhancement factor.

It is worth noting that the errors discussed in this paper are much less than observational errors in humidity values due to the hygrometers. But the use of (relatively) inaccurate formulas leads to the appearance of systematic errors in meteorological data. These errors increase the complexity of the statistical structure of the data.

#### Acknowledgements

The work of Oleg Alduchov was supported by the Resident Research Associateship of the National Research Council of the National Academy of Sciences and funded by the NCDC. The Comprehensive Aerological Research Data Set (CARDS) program is supported by the U. S. Department of Energy under contract No. DE-AI05-90ER61011. The CARDS program is also supported by the Climate and Global Change program of NOAA and the National Climatic Data Center. We thank Dr William Elliott for his careful review of this paper.



## References

- Abbott P.F., and Tabony R.C., 1985: The estimation of humidity parameters, *Meteor. Mag.*, 114, 49-56.
- Alduchov, O., 1988: On calculation accuracy of humidity characteristics, *Processing of RIHMI-WDC*, issue 147, 40-47.
- Bögel, W., 1977: Neue Näherungsgleichungen für den Sättigungsdruck des Wasserdampfes und für die in der Meteorologie gebräuchlihen ufeuchte-Parameter. *DFVLR-institut für Physik der Atmosphäre Oberpfaffenhifen*, 1977, DLR-FB, 77-52, 158 p.
- Buck, A., 1981: New equations for computing vapor pressure and enhancement factor, *J. Appl. Meteor.*, 20, 1527-1532.
- Cohen, E.R. and B.N. Taylor, 1973: Fundamental physical constants. *J. Phys. Chem.*, Ref Data 2. 663-734.
- Elliott W.P., and D.J. Gaffen, 1991: On the utility of radiosonde humidity archives for climate studies. *Bull. Amer. Meteor. Soc.*, 70, 1507-1520.
- Elliott W.P. and D.J. Gaffen, 1993: Effects of conversion algorithms on reported upper-air dewpoint depressions. *Bull. Amer. Meteor. Soc.*, 72,
- Federal Meteorological Handbook No.3. Draft. June 1991. Washington, D.C.
- Goff, J.A., and Gratch, S., 1945: Thermodynamic properties of moist air, *ASHVE Trans.*, 51, 125-157.
- Goff, J.A., and Gratch, S., 1946: Low-pressure properties of water from -160 to 212°F, *ASHVE Trans.*, 52, 95-129.
- Goff J.A., 1965: Saturation pressure of water on new Kelvin scale, In Wexler, A., (editor): *Humidity and moisture*, Vol.3. New York, Reinold Publishing Corporation, 289-292.
- Guildner, L.A., D.P. Johnson, and F.E. Jones, 1976: Vapor pressure of water at its triple point. *J. Res. Natl. Bur. Stand.*, 80A, 505-521.
- Hooper, A.H., 1986: Algorithms for automatic aerological sounding. *Instruments and observing methods. Report No. 21. WMO/TD-No. 175.*
- Matveev, L.T., 1965: *Osnovy Obshchei Meteorologii: Fizika Atmosfery, Gidrometeorologicheskoe Izadatel'stvo, Leningrad.*

- Magnus, G., 1844: Versuche uber die Spannkrafte des Wasserdampfes. *Ann. Phys. Chem. (Poggendorff)*, 61, 225.
- Murray F.W., 1966: On the computation of saturation vapor pressure, *J. Appl. Meteor.*, 5, 203-204.
- Sargent, G.P., 1980: Computation of vapor pressure, dew point and relative humidity from dry- and wet-bulb temperatures, *Meteor. Mag.*, 109, 238-246.
- Sonntag D., 1990: Important new values of the physical constants of 1986, vapor pressure formulations based on ITS-90, and psychrometer formulae. *Z. Meteorol.*, 70, 340-344.
- Sonntag D., and Heinze D., 1982: Sättigungsdampfdruck- und sättigungsdampfdichtetafeln für wasser und eis. VEB Deutscher Verlag für Grundstoffindustrie. Leipzig, 55 S.
- Stimson, H.F., 1969: Some precise measurements of the vapor pressure of water in the range from 25 to 100 C. *J. Res. Natl. Bur. Stand.*, 73A, 493-496.
- Tabata S., 1973: A simple but accurate formula for the saturation vapor pressure over liquid water, *J. Appl. Meteor.*, 12, 1410-1411.
- Tetens, O., 1930: Über einige meteorologische Begriffe. *Z. Geophys.* 6, 297-309.
- Wexler A., 1976: Vapor pressure formulation for water in the range 0°C to 100°C - A revision, *J. Res. Natl. Bur. Stand.*, 80A, 775pp.
- World Meteorological Organization, 1966: International Meteorological Tables, WMO - No.188, TP. 94.

## Appendix

Fig. 2 shows a  $d_k(a,b,c)$  surface which is very complex, the  $r_k(a,b,c)$  surface which is not shown is also complex. This surface has numerous local minima, and it is not easy to use standard mathematical methods to find a global minimum on the surface  $d_k$ . The simplest way to find coefficients  $a'$ ,  $b'$  and  $c'$ , which describe the minimum  $d_k$ , is to iterate over a dense three-dimensional set of points  $(a,b,c)$  calculating  $d_k$  and find the minimum  $(a',b',c')$ .

In order not to miss a real minimum, we need to use small increments for each step in the iteration (we used 0.001 for  $a$ , 0.01 for  $b$ , and 0.0001 for  $c$ ). However, this fine resolution produces a matrix of points which is too large and requires too much computer time to iterate over all of the points in the matrix.

We can significantly reduce the number of points  $(a,b,c)$  to iterate over if we solve a similar, but simpler task. Let's consider the expression (12) where  $e(t)$  is defined by (8). Using simple transformations we obtain the following

$$d_k = \underset{t}{\text{MAX}} \left| \frac{e^{\frac{at}{b+t} + \ln c - \ln E(t)} - 1}{dE(t)/E(t)} \right|. \quad (29)$$

It is clear that  $d_k$  will be small only if the exponent of  $e$  is close to zero. The exponential can be expanded in a Taylor series. If we assume that the exponent is small, the conditions of interest, the higher order terms of the Taylor's series expansion can be neglected. These assumptions lead to the following expression,

$$d_k = \underset{t}{\text{MAX}} \left| \frac{at/(b+t) + \ln c - \ln E(t)}{dE(t)/E(t)} \right|. \quad (30)$$

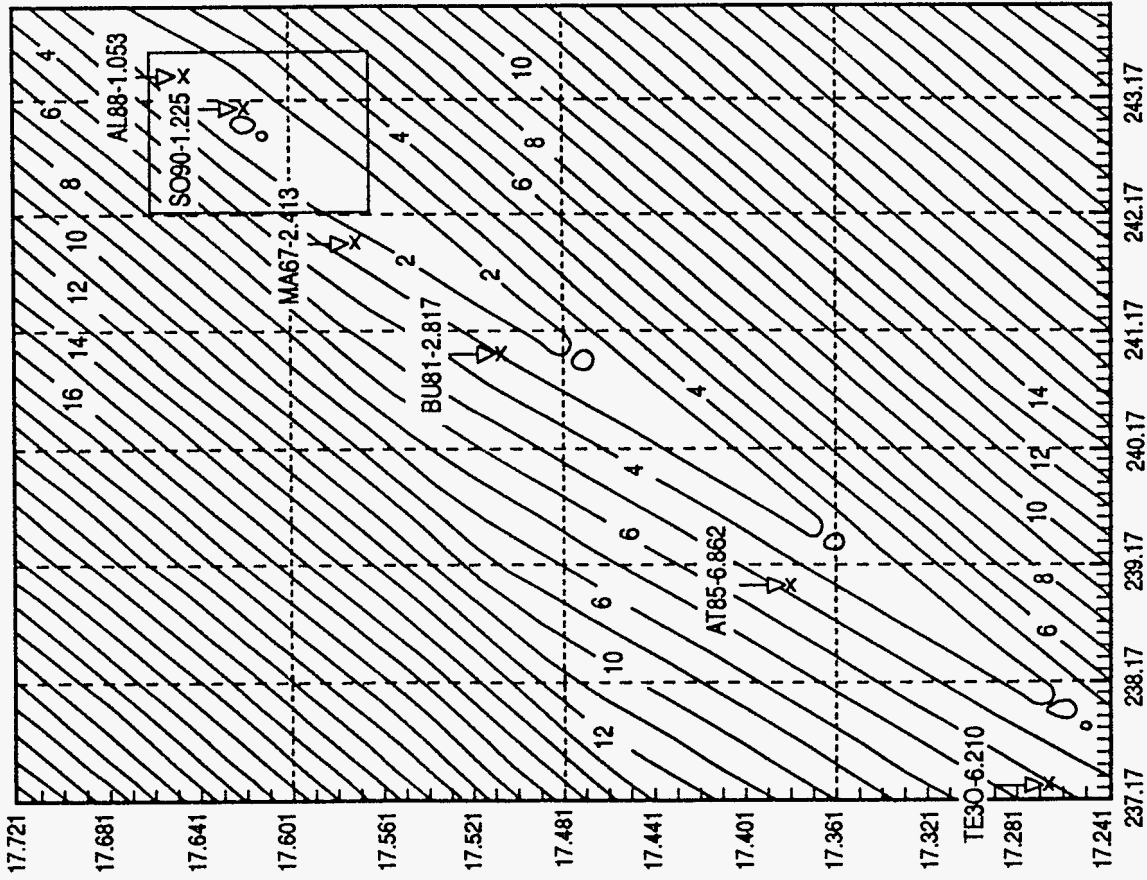
The solution is found by finding  $\min(d_k)$ . However, this is a very difficult problem to solve. The definition of the absolute value suggests an approximation to (30) which is much simpler to solve

$$\sum_t \left( \frac{at/(b+t) + \ln c - \ln E(t)}{dE(t)/E(t)} \right)^2 \rightarrow \underset{a,b,c}{\text{MIN}}. \quad (31)$$

To solve (31),  $b$  is fixed and a system of linear equations are generated and solutions are found. The solution to (31) is not exactly the solution to (30). However, experience has shown that it is a close approximation to (30), and hence to (12).

This procedure generates a three-dimensional set of points  $(a, b, c)$  used to find the minimum  $d_k$ . We can use the relationships  $a = a(b)$  and  $c = c(b)$  for each fixed  $b$  to limit the iteration to small neighborhoods of  $a$  and  $c$ .

a)



b)

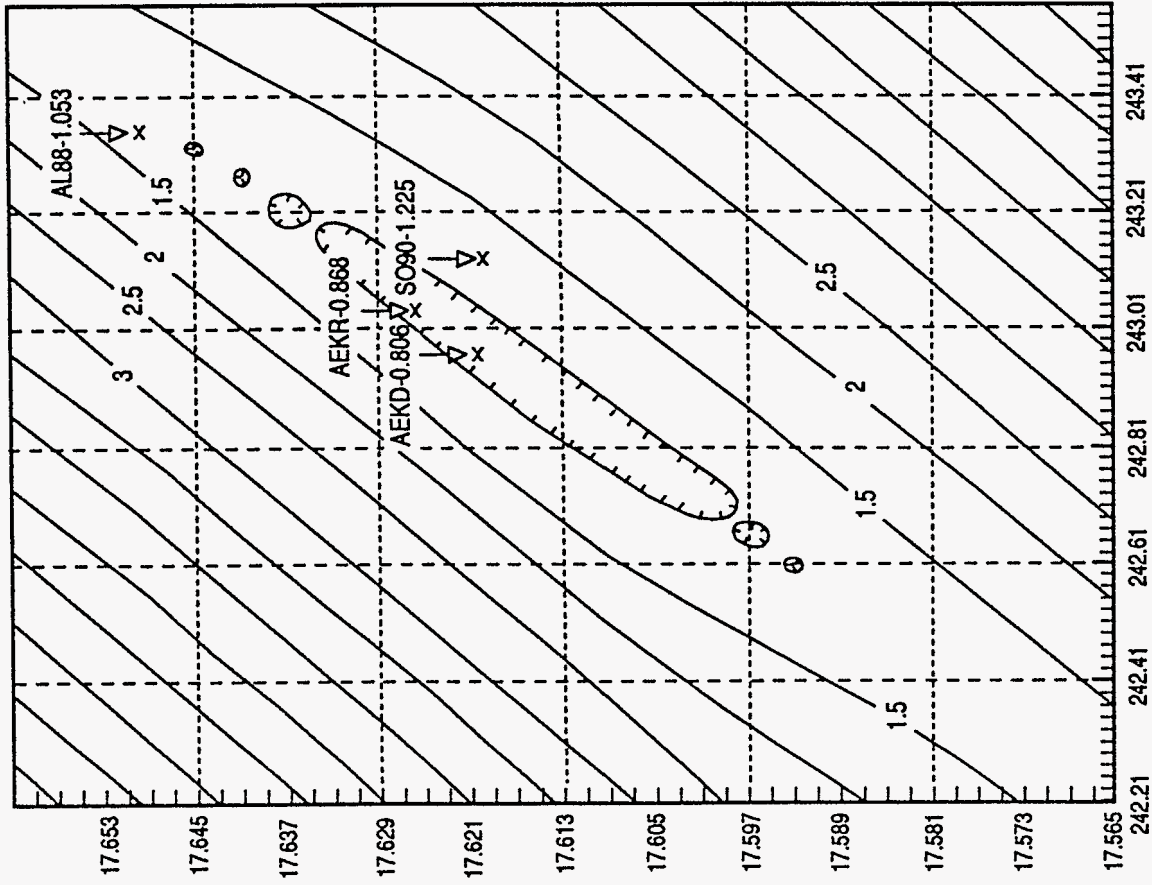


Fig. 2. Surface  $d_k$  (a, b, c = 6.1102) for the saturation vapor pressure over water.